

On superconformal Chern-Simons-matter theories in $\mathcal{N} = 4$ superspace

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Abstract

In three dimensions, every known $\mathcal{N} = 4$ supermultiplet has an off-shell completion. However, there is no off-shell $\mathcal{N} = 4$ formulation for the known extended superconformal Chern-Simons (CS) theories with eight and more supercharges. To achieve a better understanding of this issue, we provide $\mathcal{N} = 4$ superfield realisations for the equations of motion which correspond to various $\mathcal{N} = 4$ and $\mathcal{N} = 6$ superconformal CS theories, including the Gaiotto-Witten theory and the ABJM theory. These superfield realisations demonstrate that the superconformal CS theories with $\mathcal{N} \geq 4$ (except for the Gaiotto-Witten theory) require a reducible long $\mathcal{N} = 4$ vector multiplet, from which the standard left and right $\mathcal{N} = 4$ vector multiplets are obtained by constraining the field strength to be either self-dual or anti self-dual. Such a long multiplet naturally originates upon reduction of any off-shell $\mathcal{N} > 4$ vector multiplet to $\mathcal{N} = 4$ superspace. For the long $\mathcal{N} = 4$ vector multiplet we develop a prepotential formulation. It makes use of two prepotentials being subject to the constraint which defines the so-called hybrid projective multiplets introduced in the framework of $\mathcal{N} = 4$ supergravity-matter systems in arXiv:1101.4013. We also couple $\mathcal{N} = 4$ superconformal CS theories to $\mathcal{N} = 4$ conformal supergravity.

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1 Introduction

Since the 2004 work by Schwarz [1], much progress has been achieved in the construction of extended superconformal Chern-Simons-matter (CS) theories in three dimensions (3D). The famous CS theories with $\mathcal{N} = 8$ [2, 3, 4], $\mathcal{N} = 6$ [5, 6, 7] and $\mathcal{N} = 4$ [8, 9] supersymmetry have been proposed.

All superconformal CS theories with $\mathcal{N} > 3$ can be realised as special off-shell $\widehat{\mathcal{N}}$ -extended Chern-Simons-matter systems, where $\widehat{\mathcal{N}} \leq 3$. In such realisations in terms of $\widehat{\mathcal{N}}$ -extended superfields, $\mathcal{N} - \widehat{\mathcal{N}}$ supersymmetries are hidden. Of course, the $\widehat{\mathcal{N}} = 3$ realisation [10] is the most powerful, since it allows one to keep manifest the maximal amount of supersymmetry. The special feature of the three cases $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 3$ is that the off-shell supersymmetric *pure* CS action exists for any gauge group. However, no $\mathcal{N} \geq 4$ supersymmetric CS action can be constructed (for a recent proof, see [11]), although abelian $\mathcal{N} = 4$ BF couplings are abundant [12]. In this regard, especially paradoxical is the situation with $\mathcal{N} = 4$ supersymmetry. Every 3D $\mathcal{N} = 4$ supermultiplet admits an off-shell realisation. There exist off-shell formulations for various 3D $\mathcal{N} = 4$ supersymmetric theories, including the Yang-Mills theories with Poincaré [13, 14] and anti-de Sitter supersymmetry [15], the most general σ -models with Poincaré [16] and anti-de Sitter supersymmetry [17], and general supergravity-matter systems [18]. However, it is impossible to construct a $\mathcal{N} = 4$ supersymmetric CS action, at least in terms of the standard vector multiplets and hypermultiplets.

Since there is no way to realise the known $\mathcal{N} \geq 4$ superconformal CS theories in terms of $\mathcal{N} = 4$ superfields off the mass shell, in this note we would like to analyse a simpler problem. We will only formulate the equations of motion for $\mathcal{N} \geq 4$ superconformal CS theories in $\mathcal{N} = 4$ superspace. Similar on-shell realisations in $\mathcal{N} = 6$ and $\mathcal{N} = 8$ superspaces have been given in [19, 20, 21].

This note is organised as follows. In section 2 we consider general $\mathcal{N} = 4$ superconformal CS theories and show that the hypermultiplet equations of motion require consistency conditions which impose non-trivial constraints on the gauge group and its representation to which the hypermultiplet belongs. In sections 3 and 4 we present the $\mathcal{N} = 4$ superfield equations of motion for the Gaiotto-Witten and ABJM theories, respectively. For these models we also construct their supercurrents and other conserved current multiplets. Section 5 is devoted to the prepotential formulation for the large $\mathcal{N} = 4$ vector multiplet. In conclusion, we discuss the structure of the long $\mathcal{N} = 4$ vector multiplet from $\mathcal{N} = 3$ superspace perspective. In appendix A we give a proof that the constraints on the gauge group derived in section 2 are equivalent

to the fundamental identity of Gaiotto and Witten [8]. In appendix B we review the structure of the \mathcal{N} -extended vector multiplet coupled to conformal supergravity.

2 $\mathcal{N} = 4$ superconformal CS theories

The $\mathcal{N} = 4$ Minkowski superspace can be parametrised by coordinates $z^{\mathcal{A}} = (x^{\alpha\beta}, \theta_{\tilde{i}}^{\alpha})$. Here $x^{\alpha\beta} = x^{(\alpha\beta)}$ are the bosonic coordinates, where $\alpha, \beta = 1, 2$ are spinor indices.¹ The Grassmann coordinates $\theta_{\tilde{i}}^{\alpha}$ carry two isospinor indices, $i = 1, 2$ and $\tilde{i} = 1, 2$, which correspond to the subgroups $\text{SU}(2)_{\text{L}}$ and $\text{SU}(2)_{\text{R}}$ of the $\mathcal{N} = 4$ R -symmetry group $\text{SU}(2)_{\text{L}} \times \text{SU}(2)_{\text{R}}$. The spinor covariant derivatives $D_{\alpha}^{\tilde{i}\tilde{j}}$ satisfy the anti-commutation relations $\{D_{\alpha}^{\tilde{i}\tilde{j}}, D_{\beta}^{j\tilde{j}}\} = 2i\varepsilon^{ij}\varepsilon^{\tilde{i}\tilde{j}}\partial_{\alpha\beta}$.

To describe a non-abelian $\mathcal{N} = 4$ vector multiplet, we introduce gauge covariant derivatives $\mathcal{D}_{\mathcal{A}} = (\mathcal{D}_{\alpha\beta}, \mathcal{D}_{\alpha}^{\tilde{i}\tilde{j}}) = D_{\mathcal{A}} + iV_{\mathcal{A}}$, where $D_{\mathcal{A}} = (\partial_{\alpha\beta}, D_{\alpha}^{\tilde{i}\tilde{j}})$ denotes the covariant derivatives of $\mathcal{N} = 4$ superspace. The gauge connection $V_{\mathcal{A}}$ takes its values in the Lie algebra of the gauge group G . Given a matter multiplet Φ belonging to some representation of the gauge group, the gauge transformation laws of $V_{\mathcal{A}}$ and Φ are as follows:

$$\mathcal{D}'_{\mathcal{A}} = e^{i\tau}\mathcal{D}_{\mathcal{A}}e^{-i\tau} , \quad (2.1a)$$

$$\Phi' = e^{i\tau}\Phi . \quad (2.1b)$$

Here the Lie-algebra-valued gauge parameter $\tau(z)$ is Hermitian, $\tau^{\dagger} = \tau$, but otherwise unconstrained.

In 3D \mathcal{N} -extended supersymmetry, one can impose a universally looking constraint to describe a vector multiplet, see Appendix B. In the $\mathcal{N} = 4$ case the constraint amounts to

$$\{\mathcal{D}_{\alpha}^{\tilde{i}\tilde{j}}, \mathcal{D}_{\beta}^{j\tilde{j}}\} = 2i\varepsilon^{ij}\varepsilon^{\tilde{i}\tilde{j}}\mathcal{D}_{\alpha\beta} + \varepsilon_{\alpha\beta}\varepsilon^{ij}\mathcal{W}^{\tilde{i}\tilde{j}} + \varepsilon_{\alpha\beta}\varepsilon^{\tilde{i}\tilde{j}}\mathcal{W}^{ij} . \quad (2.2)$$

The field strengths $\mathcal{W}^{ij} = \mathcal{W}^{(ij)}$ and $\mathcal{W}^{\tilde{i}\tilde{j}} = \mathcal{W}^{(\tilde{i}\tilde{j})}$ are Hermitian in the sense that $(\mathcal{W}^{ij})^{\dagger} = \mathcal{W}_{ij} = \varepsilon_{ik}\varepsilon_{jl}\mathcal{W}^{kl}$, and similarly for $\mathcal{W}^{\tilde{i}\tilde{j}}$. They are subject to the Bianchi identities

$$\mathcal{D}_{\alpha}^{\tilde{i}(i}\mathcal{W}^{jk)} = 0 , \quad (2.3a)$$

$$\mathcal{D}_{\alpha}^{i(\tilde{i}}\mathcal{W}^{\tilde{j}k)} = 0 . \quad (2.3b)$$

¹The variables $x^{\alpha\beta}$ are related to the coordinates x^m of Minkowski space \mathbb{M}^3 by the rule $x^{\alpha\beta} = x^m(\gamma_m)^{\alpha\beta}$, with $(\gamma)^{\alpha\beta}$ the gamma-matrices with upper spinor indices. The partial derivatives $\partial_{\alpha\beta}$ are defined similarly, $\partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial_m$, such that $\partial_{\alpha\beta}x^{\gamma\delta} = -2\delta_{\alpha}^{(\gamma}\delta_{\beta}^{\delta)}$. Our two-component spinor notation and conventions, including the definition of the gamma-matrices, follow [16, 18].

The vector multiplet described by the constraint (2.2) is reducible and, therefore, will be called “long.” There exist two irreducible off-shell $\mathcal{N} = 4$ vector multiplets [12, 13, 14], which are obtained from (2.2) by imposing additional constraints. Following the terminology of [18], the left vector multiplet is subject to the additional constraint

$$\mathcal{W}^{\tilde{i}\tilde{j}} = 0 . \quad (2.4a)$$

The right vector multiplet is obtained by setting

$$\mathcal{W}^{ij} = 0 . \quad (2.4b)$$

In what follows, we will work with the long vector multiplet (2.2) due to the following two reasons: (i) as will be shown below, it naturally corresponds to the $\mathcal{N} \geq 4$ superconformal CS theories; and (ii) it is obtained by reducing the off-shell $\mathcal{N} > 4$ vector multiplets to $\mathcal{N} = 4$ superspace.

Similar to the left and right vector multiplets, there are two inequivalent $\mathcal{N} = 4$ hypermultiplets, left and right ones. In this paper, we will be interested in *on-shell* hypermultiplets. The left hypermultiplet is described by a left isospinor $q^i = (q_a^i)$ (which is viewed in this section as a column vector) and its conjugate $\bar{q}_i = (q^i)^\dagger$. The right hypermultiplet is described by a right isospinor $\tilde{q}^{\tilde{i}} = (q_{\tilde{a}}^{\tilde{i}})$ and its conjugate $\bar{\tilde{q}}_{\tilde{i}} = (\tilde{q}^{\tilde{i}})^\dagger$. Both the left and right hypermultiplets are assumed to interact with the long vector multiplet. In general they belong to different representations of the gauge group G , with the generators $(T_A)_a{}^b$ and $(\tilde{T}_A)_{\tilde{a}}{}^{\tilde{b}}$, respectively. The hypermultiplet equations of motion have the form

$$\mathcal{D}_\alpha^{\tilde{i}(i} q^{j)} = 0 , \quad \mathcal{D}_\alpha^{\tilde{i}(i} \bar{q}^{j)} = 0 , \quad (2.5a)$$

$$\mathcal{D}_\alpha^{i(\tilde{i}} q^{\tilde{j})} = 0 , \quad \mathcal{D}_\alpha^{i(\tilde{i}} \bar{\tilde{q}}^{\tilde{j})} = 0 , \quad (2.5b)$$

and are similar to the constraints introduced by Sohnius [22] to describe the $\mathcal{N} = 2$ hypermultiplet in four dimensions. The crucial difference of these equations from their $\mathcal{N} = 3$ counterparts is that they require the following consistency conditions

$$\mathcal{W}^{(ij} q^{k)} = 0 , \quad \mathcal{W}^{(\tilde{i}\tilde{j}} \bar{\tilde{q}}^{\tilde{k})} = 0 . \quad (2.6)$$

In the case of $\mathcal{N} = 3$ superconformal CS theories with matter, no restriction on the gauge group and its representation occur, see [10] for more details.

Up to now, our consideration was completely general. In what follows we restrict ourselves to superconformal theories. In this case the gauge multiplet cannot have independent degrees of freedom as the Yang-Mills coupling is not permitted. On the

equations of motion the field strengths should be expressed in terms of hypermultiplets. A large family of superconformal theories is described by the equations of motion for the field strengths which are bilinear in hypermultiplets

$$\mathcal{W}_A^{ij} = \mathrm{i}\kappa g_{AB} \bar{q}^{(i} T^B q^{j)} , \quad (2.7a)$$

$$\mathcal{W}_A^{\tilde{i}\tilde{j}} = \mathrm{i}\tilde{\kappa} g_{AB} \bar{q}^{(\tilde{i}} \tilde{T}^B q^{\tilde{j})} , \quad (2.7b)$$

where κ and $\tilde{\kappa}$ are some dimensionless coefficients and g_{AB} is an invariant quadratic form on the Lie algebra of the gauge group G . The consistency conditions (2.6) lead to the following equations

$$\bar{q}^{a(i} q_b^j q_c^{k)} g_{AB} (T^A)_a{}^b (T^B)_d{}^c = 0 , \quad (2.8a)$$

$$\bar{q}^{\tilde{a}(\tilde{i}} q_{\tilde{b}}^{\tilde{j}} q_{\tilde{c}}^{\tilde{k})} g_{AB} (\tilde{T}^A)_{\tilde{a}}{}^{\tilde{b}} (\tilde{T}^B)_{\tilde{d}}{}^{\tilde{c}} = 0 . \quad (2.8b)$$

These equations require the generators to obey the relations

$$g_{AB} (T^A)_a{}^b (T^B)_d{}^c = 0 , \quad (2.9a)$$

$$g_{AB} (\tilde{T}^A)_{\tilde{a}}{}^{\tilde{b}} (\tilde{T}^B)_{\tilde{d}}{}^{\tilde{c}} = 0 , \quad (2.9b)$$

which are strong constraints on the possible gauge group G and its representations. These relations are, in fact, equivalent to the fundamental identity for the generators of the gauge group derived in [8] (see Appendix A for the proof).

The dynamical system under consideration is characterised by the supercurrent (compare with [23])

$$J = \bar{q}_i q^i - \bar{q}_{\tilde{i}} q^{\tilde{i}} , \quad (2.10)$$

which obeys the conservation equation [24, 11, 23]

$$D^\alpha (\bar{i}^i D_\alpha^j \bar{j}) J = 0 , \quad (2.11)$$

as a consequence of the equations of motion (2.5) and (2.7).

3 The equations of motion for the Gaiotto-Witten theory

In the previous section we have provided the $\mathcal{N} = 4$ superfield description for the general $\mathcal{N} = 4$ superconformal CS theories studied in [9]. The Gaiotto-Witten theory [8] is a special member of this family. This theory has only one type of hypermultiplets, q^i , and no right hypermultiplets, $q^{\tilde{i}} = 0$. Then we should also have

$\mathcal{W}^{\tilde{i}\tilde{j}} = 0$, as a consequence of (2.7b), and the vector multiplet becomes short, the left one. The remaining superfields q^i and \mathcal{W}^{ij} obey the equations of motion (2.5a) and (2.7a). Now we will show that the constraint (2.9a) is satisfied for the Gaiotto-Witten theory [8].

This theory has two field strengths $\mathcal{W}_{\mathfrak{L}}^{ij}$ and $\mathcal{W}_{\mathfrak{R}}^{ij}$ associated with a gauge group of the form $G = G_{\mathfrak{L}} \times G_{\mathfrak{R}}$ that possesses a representation compatible with eq. (2.9a) (see also the discussion in Appendix A). One admissible choice is $G = \mathrm{U}(M) \times \mathrm{U}(N)$, and the hypermultiplet transforms in the bi-fundamental representation of G . Only this case is considered in the present section. The gauge transformation laws of these superfields are

$$q'^i = e^{i\tau_{\mathfrak{L}}} q^i e^{-i\tau_{\mathfrak{R}}} , \quad \bar{q}'^i = e^{i\tau_{\mathfrak{R}}} \bar{q}^i e^{-i\tau_{\mathfrak{L}}} , \quad (3.1a)$$

$$\mathcal{W}'_{\mathfrak{L}}{}^{ij} = e^{i\tau_{\mathfrak{L}}} \mathcal{W}_{\mathfrak{L}}^{ij} e^{-i\tau_{\mathfrak{L}}} , \quad \mathcal{W}'_{\mathfrak{R}}{}^{ij} = e^{i\tau_{\mathfrak{R}}} \mathcal{W}_{\mathfrak{R}}^{ij} e^{-i\tau_{\mathfrak{R}}} \quad (3.1b)$$

where the gauge parameters $\tau_{\mathfrak{L}}(z)$ and $\tau_{\mathfrak{R}}(z)$ are Hermitian matrices taking their values in the Lie algebras of the gauge groups $G_{\mathfrak{L}}$ and $G_{\mathfrak{R}}$, respectively.

The equations of motion for the hypermultiplets (2.5a) and the vector multiplet (2.7a) become

$$\mathcal{D}_{\alpha}^{\tilde{i}(i} q^{j)} = 0 , \quad \mathcal{D}_{\alpha}^{\tilde{i}(i} \bar{q}^{j)} = 0 , \quad (3.2a)$$

$$\mathcal{W}_{\mathfrak{L}}^{ij} = i\kappa q^{(i} \bar{q}^{j)} , \quad \mathcal{W}_{\mathfrak{R}}^{ij} = i\kappa \bar{q}^{(i} q^{j)} . \quad (3.2b)$$

Here the covariant derivatives act on the hypermultiplet superfields by the rule

$$\mathcal{D}_{\alpha}^{\tilde{i}i} q^j = D_{\alpha}^{\tilde{i}i} q^j + iV_{\mathfrak{L}\alpha}^{\tilde{i}i} q^j - i q^j V_{\mathfrak{R}\alpha}^{\tilde{i}i} , \quad \mathcal{D}_{\alpha}^{\tilde{i}i} \bar{q}^j = D_{\alpha}^{\tilde{i}i} \bar{q}^j + iV_{\mathfrak{R}\alpha}^{\tilde{i}i} \bar{q}^j - i \bar{q}^j V_{\mathfrak{L}\alpha}^{\tilde{i}i} , \quad (3.3)$$

in accordance with the transformation laws (3.1a). For the equations of motion (3.2a), the consistency conditions (2.6) take the form

$$\mathcal{W}_{\mathfrak{L}}^{(ij} q^{k)} - q^{(k} \mathcal{W}_{\mathfrak{R}}^{ij)} = 0 , \quad \mathcal{W}_{\mathfrak{R}}^{(ij} \bar{q}^{k)} - \bar{q}^{(k} \mathcal{W}_{\mathfrak{L}}^{ij)} = 0 \quad (3.4)$$

and are identically satisfied for the superfield strengths (3.2b).

In concluding this section we construct the $\mathcal{N} = 4$ supercurrent J and $\mathrm{U}(1)$ flavour current multiplet L^{ij} in the Gaiotto-Witten theory

$$J = \mathrm{tr} (q^i \bar{q}_i) , \quad L^{ij} = i \mathrm{tr} (q^{(i} \bar{q}^{j)}) , \quad (3.5)$$

which obey the conservation equations (see [23] for more details)

$$D^{\alpha}{}^{\tilde{i}(i} D_{\alpha}^{\tilde{j})j} J = 0 , \quad D^{\tilde{i}(i} L^{j)k} = 0 , \quad (3.6)$$

as a consequence of the equations of motion (3.2). Of course, the flavour current multiplet is non-trivial only for the gauge group which possesses the $\mathrm{U}(1)$ factor. The two- and three-point correlation functions of the supercurrent and flavour current multiplets in general $\mathcal{N} = 4$ superconformal field theories were studied in [23].

4 The equations of motion for the ABJM theory

Before presenting our $\mathcal{N} = 4$ superfield realisation for the $\mathcal{N} = 6$ superconformal CS theory proposed in [5] and known as the ABJM theory, we would like to make some preliminary comments. In three dimensions, the \mathcal{N} -extended vector multiplet can be formulated in \mathcal{N} -extended superspace and is off-shell [21] (see also [25]). A brief review of the \mathcal{N} -extended vector multiplet coupled to conformal supergravity is given in Appendix B. In the flat case, every $\mathcal{N} > 4$ vector multiplet can be reduced to $\mathcal{N} = 4$ Minkowski superspace, resulting in the long $\mathcal{N} = 4$ vector multiplet coupled to several additional constrained superfields. In particular, it can be shown that the $\mathcal{N} = 6 \rightarrow \mathcal{N} = 4$ reduction for the field strength $\mathcal{W}^{IJ} = -\mathcal{W}^{JI}$ of the $\mathcal{N} = 6$ vector multiplet² (with I, J being $\text{SO}(6)$ indices, see Appendix B) leads to the following $\mathcal{N} = 4$ superfields

$$\mathcal{W}^{ij} , \quad \mathcal{W}^{\tilde{i}\tilde{j}} , \quad \mathcal{Y}^{\tilde{i}\tilde{i}} , \quad \mathcal{Z}^{\tilde{i}\tilde{i}} , \quad \mathcal{S} . \quad (4.1)$$

It may be shown that the $\mathcal{N} = 6$ Bianchi identity (B.6) is equivalent to the following constraints on the above $\mathcal{N} = 4$ superfields:

$$\mathcal{D}_\alpha^{\tilde{i}(i} \mathcal{W}^{jk)} = 0 , \quad \mathcal{D}_\alpha^{i(\tilde{i}} \mathcal{W}^{\tilde{j}k)} = 0 , \quad (4.2a)$$

$$\mathcal{D}_\alpha^{\tilde{i}(i} \mathcal{Y}^{j)\tilde{j}} = 0 , \quad \mathcal{D}_\alpha^{\tilde{i}(i} \mathcal{Z}^{j)\tilde{j}} = 0 , \quad (4.2b)$$

$$\mathcal{D}^{\alpha(\tilde{i}(i} \mathcal{D}_\alpha^{j)\tilde{j}} \mathcal{S} = [\mathcal{Y}^{\tilde{i}(i} , \mathcal{Z}^{j)\tilde{j}}] . \quad (4.2c)$$

The constraints (4.2b) tell us that $\mathcal{Y}^{\tilde{i}\tilde{i}}$ and $\mathcal{Z}^{\tilde{i}\tilde{i}}$ are examples of the so-called hybrid supermultiplets introduced for the first time in [18] in the framework of general $\mathcal{N} = 4$ supergravity-matter systems. Eq. (4.2c) may be interpreted as the condition that \mathcal{S} is a hybrid linear superfield.

Unlike the theory studied in section 3, now we consider the case when both hypermultiplets q^i and $\tilde{q}^{\tilde{i}}$ have non-trivial dynamics. Here we assume that the hypermultiplets transform in the bi-fundamental representation of the gauge group $G = G_\mathfrak{L} \times G_\mathfrak{R} = \text{U}(M) \times \text{U}(N)$,

$$q'^i = e^{i\tau_\mathfrak{L}} q^i e^{-i\tau_\mathfrak{R}} , \quad \tilde{q}'^{\tilde{i}} = e^{i\tau_\mathfrak{R}} \tilde{q}^{\tilde{i}} e^{-i\tau_\mathfrak{L}} , \quad (4.3a)$$

$$q^{\tilde{i}} = e^{i\tau_\mathfrak{L}} q^{\tilde{i}} e^{-i\tau_\mathfrak{R}} , \quad \tilde{q}^i = e^{i\tau_\mathfrak{R}} \tilde{q}^i e^{-i\tau_\mathfrak{L}} , \quad (4.3b)$$

where the gauge superfield parameters $\tau_\mathfrak{L}(z)$ and $\tau_\mathfrak{R}(z)$ are Hermitian and otherwise unconstrained.

²Using the isomorphism $\text{SU}(4) \cong \text{SO}(6)/\mathbb{Z}_2$, the $\mathcal{N} = 6$ vector multiplet can be described in the $\text{SU}(4)$ notation [20].

Due to the structure of the gauge group, $G = G_{\mathfrak{L}} \times G_{\mathfrak{R}}$, there are two long $\mathcal{N} = 4$ vector multiplets and the corresponding field strengths. We have the field strengths $\mathcal{W}_{\mathfrak{L}}^{ij}$ and $\mathcal{W}_{\mathfrak{L}}^{\tilde{i}\tilde{j}}$ which take values in the Lie algebra of the gauge group $G_{\mathfrak{L}}$ and similar ones, $\mathcal{W}_{\mathfrak{R}}^{ij}$ and $\mathcal{W}_{\mathfrak{R}}^{\tilde{i}\tilde{j}}$, which correspond to the gauge group $G_{\mathfrak{R}}$. They transform in the adjoint representations of these groups

$$\mathcal{W}_{\mathfrak{L}}'^{ij} = e^{i\tau_{\mathfrak{L}}} \mathcal{W}_{\mathfrak{L}}^{ij} e^{-i\tau_{\mathfrak{L}}} , \quad \mathcal{W}_{\mathfrak{L}}'^{\tilde{i}\tilde{j}} = e^{i\tau_{\mathfrak{L}}} \mathcal{W}_{\mathfrak{L}}^{\tilde{i}\tilde{j}} e^{-i\tau_{\mathfrak{L}}} , \quad (4.4a)$$

$$\mathcal{W}_{\mathfrak{R}}'^{ij} = e^{i\tau_{\mathfrak{R}}} \mathcal{W}_{\mathfrak{R}}^{ij} e^{-i\tau_{\mathfrak{R}}} , \quad \mathcal{W}_{\mathfrak{R}}'^{\tilde{i}\tilde{j}} = e^{i\tau_{\mathfrak{R}}} \mathcal{W}_{\mathfrak{R}}^{\tilde{i}\tilde{j}} e^{-i\tau_{\mathfrak{R}}} . \quad (4.4b)$$

The natural generalisation of the equations of motion (3.2) reads

$$\mathcal{D}_{\alpha}^{\tilde{i}(i} q^{j)} = 0 , \quad \mathcal{D}_{\alpha}^{i(\tilde{i}} q^{\tilde{j})} = 0 , \quad (4.5a)$$

$$\mathcal{W}_{\mathfrak{L}}^{ij} = i\kappa q^{(i} \bar{q}^{j)} , \quad \mathcal{W}_{\mathfrak{R}}^{ij} = i\kappa \bar{q}^{(i} q^{j)} , \quad \mathcal{W}_{\mathfrak{L}}^{\tilde{i}\tilde{j}} = i\kappa q^{(\tilde{i}} \bar{q}^{\tilde{j})} , \quad \mathcal{W}_{\mathfrak{R}}^{\tilde{i}\tilde{j}} = i\kappa \bar{q}^{(\tilde{i}} q^{\tilde{j})} . \quad (4.5b)$$

The consistency conditions (2.9) are automatically satisfied for these equations,

$$\mathcal{W}_{\mathfrak{L}}^{(ij} q^{k)} - q^{(k} \mathcal{W}_{\mathfrak{R}}^{ij)} = 0 , \quad \mathcal{W}_{\mathfrak{L}}^{(\tilde{i}\tilde{j}} q^{\tilde{k})} - q^{(\tilde{k}} \mathcal{W}_{\mathfrak{R}}^{\tilde{i}\tilde{j})} = 0 . \quad (4.6)$$

The ABJM theory is $\mathcal{N} = 6$ superconformal. Therefore, there should exist hypermultiplet composites that realise the superfields $\mathcal{Y}^{\tilde{i}\tilde{i}}$, $\mathcal{Z}^{\tilde{i}\tilde{i}}$ and \mathcal{S} in (4.1) on the mass shell. Since we have two gauge groups, $G_{\mathfrak{L}}$ and $G_{\mathfrak{R}}$, the number of superfields (4.1) is doubled. We will distinguish them by attaching the subscripts \mathfrak{L} and \mathfrak{R} to them. It is clear that they should be expressed in terms of the hypermultiplet superfields. Indeed, the expressions for \mathcal{W}^{ij} and $\mathcal{W}^{\tilde{i}\tilde{j}}$ are given by (4.5b). For the remaining superfields we find

$$\mathcal{Y}_{\mathfrak{L}}^{\tilde{i}\tilde{i}} = 2i\kappa(q^i \bar{q}^{\tilde{i}} + \bar{q}^{\tilde{i}} q^i) , \quad \mathcal{Y}_{\mathfrak{R}}^{\tilde{i}\tilde{i}} = 2i\kappa(\bar{q}^{\tilde{i}} q^{\tilde{i}} + q^{\tilde{i}} \bar{q}^{\tilde{i}}) , \quad (4.7a)$$

$$\mathcal{Z}_{\mathfrak{L}}^{\tilde{i}\tilde{i}} = 2\kappa(q^i \bar{q}^{\tilde{i}} - \bar{q}^{\tilde{i}} q^i) , \quad \mathcal{Z}_{\mathfrak{R}}^{\tilde{i}\tilde{i}} = 2\kappa(\bar{q}^{\tilde{i}} q^{\tilde{i}} - q^{\tilde{i}} \bar{q}^{\tilde{i}}) , \quad (4.7b)$$

$$\mathcal{S}_{\mathfrak{L}} = \kappa(q^i \bar{q}_i - \bar{q}_i q^i) , \quad \mathcal{S}_{\mathfrak{R}} = \kappa(\bar{q}_i q^i - q_i \bar{q}^i) . \quad (4.7c)$$

These superfields do satisfy the $\mathcal{N} = 6$ Bianchi identities (4.2b) and (4.2c) on the hypermultiplet equations of motion (4.5a).

Since the ABJM theory is $\mathcal{N} = 6$ superconformal, it should possess a number of conserved currents which form the $\mathcal{N} = 6$ supercurrent multiplet $J^{IJ} = -J^{JI}$ [25, 24, 26]. Upon reduction to $\mathcal{N} = 4$ superspace, the $\mathcal{N} = 6$ supercurrent may be shown to lead to the following constrained $\mathcal{N} = 4$ multiplets: (i) the $\mathcal{N} = 4$ supercurrent J ; (ii) two U(1) flavour current multiplets L^{ij} and $L^{\tilde{i}\tilde{j}}$; two SO(4) vectors $A^{\tilde{i}\tilde{i}}$ and $B^{\tilde{i}\tilde{i}}$. In the the ABJM theory, these multiplets should be given as hypermultiplet composites. Their explicit form is as follows:

- the $\mathcal{N} = 4$ supercurrent

$$J = \text{tr}(q^i \bar{q}_i) - \text{tr}(\tilde{q}^{\tilde{i}} \bar{\tilde{q}}_{\tilde{i}}) \quad (4.8)$$

obeying the conservation equation (2.11);

- the $U(1)$ flavour current multiplets

$$L^{ij} = i \text{tr}(q^{(i} \bar{q}^{j)}) , \quad L^{\tilde{i}\tilde{j}} = i \text{tr}(q^{\tilde{i}} \bar{\tilde{q}}^{\tilde{j}}) \quad (4.9)$$

obeying the conservation laws

$$D_{\alpha}^{\tilde{i}(i} L^{j k)} = 0 , \quad D_{\alpha}^{i(\tilde{i}} L^{\tilde{j} \tilde{k})} = 0 ; \quad (4.10)$$

- the $SO(4)$ vectors

$$A^{i\tilde{i}} = i \text{tr}(q^i \bar{\tilde{q}}^{\tilde{i}}) + i \text{tr}(\bar{q}^{\tilde{i}} q^i) , \quad B^{\tilde{i}i} = \text{tr}(q^i \bar{\tilde{q}}^{\tilde{i}}) - \text{tr}(\bar{\tilde{q}}^{\tilde{i}} q^i) , \quad (4.11)$$

which obey the same conservation equations

$$D_{\alpha}^{\tilde{i}(i} A^{j \tilde{j})} = 0 , \quad D_{\alpha}^{\tilde{i}(i} B^{j \tilde{j})} = 0 . \quad (4.12)$$

The hypermultiplet composites (4.8), (4.9) and (4.11) are the components of the $\mathcal{N} = 6$ supercurrent. They are conserved as a consequence of the equations of motion (4.5). It is interesting to note that these objects are obtained from the composites in (4.5b) and (4.7) by taking the matrix trace.

5 Prepotentials for the long $\mathcal{N} = 4$ vector multiplet

For the left and right $\mathcal{N} = 4$ Yang-Mills supermultiplets, there exist *prepotential formulations*. The harmonic superspace formulation was given by Zupnik [13, 14] in the case of $\mathcal{N} = 4$ Poincaré supersymmetry. The projective superspace formulation was given in [15] for the left and right Yang-Mills supermultiplets coupled to $\mathcal{N} = 4$ conformal supergravity (the case of abelian vector multiplets was described in [18]). In this section we present a prepotential formulation for the long vector multiplet as a natural generalisation of Zupnik's construction [13, 14]. A prepotential formulation for the long vector multiplet coupled to $\mathcal{N} = 4$ conformal supergravity may be obtained as a natural generalisation of the formulation developed in [15], but we will not elaborate on this here.

Let u_i^{\pm} and $v_{\tilde{i}}^{\pm}$ be standard harmonic variables for the $SU(2)_L$ and $SU(2)_R$,

$$u^{+i} u_j^{-} - u^{-i} u_j^{+} = \delta_j^i , \quad \overline{u^{+i}} = u_i^{-} , \quad (5.1a)$$

$$v^{+\tilde{i}} v_{\tilde{j}}^{-} - v^{-\tilde{i}} v_{\tilde{j}}^{+} = \delta_{\tilde{j}}^{\tilde{i}} , \quad \overline{v^{+\tilde{i}}} = v_{\tilde{i}}^{-} . \quad (5.1b)$$

The harmonics carry the labels \pm which correspond to charges with respect to certain $U(1)$ subgroups of $SU(2)_L$ and $SU(2)_R$, respectively. We will use these harmonics to parametrise smooth superfields on the harmonic superspace

$$\mathbb{M}^{3|8} \times [SU(2)/U(1)]_L \times [SU(2)/U(1)]_R , \quad (5.2)$$

Any superfield $\Phi^{(p,q)}(z, u^\pm, v^\pm)$ defined on this superspace is labeled by two integer $U(1)$ charges p and q defined by $\Phi^{(p,q)}(z, e^{\pm i\alpha} u^\pm, e^{\pm i\beta} v^\pm) = e^{i(p\alpha+q\beta)} \Phi^{(p,q)}(z, u^\pm, v^\pm)$, for real parameters α and β . It is useful to introduce left invariant vector fields for the groups $SU(2)_L$ and $SU(2)_R$

$$D^{(2,0)} = u_i^+ \frac{\partial}{\partial u_i^-} , \quad D^{(-2,0)} = u_i^- \frac{\partial}{\partial u_i^+} , \quad D^{(0,0)} = u_i^+ \frac{\partial}{\partial u_i^+} - u_i^- \frac{\partial}{\partial u_i^-} ; \quad (5.3a)$$

$$D^{(0,2)} = v_i^+ \frac{\partial}{\partial v_i^-} , \quad D^{(0,-2)} = v_i^- \frac{\partial}{\partial v_i^+} , \quad \tilde{D}^{(0,0)} = v_i^+ \frac{\partial}{\partial v_i^+} - v_i^- \frac{\partial}{\partial v_i^-} . \quad (5.3b)$$

The operators within each of these sets obey the standard $SU(2)$ commutation relations

$$[D^{(0,0)}, D^{(\pm 2,0)}] = \pm 2D^{(\pm 2,0)} , \quad [D^{(2,0)}, D^{(-2,0)}] = D^{(0,0)} ; \quad (5.4a)$$

$$[\tilde{D}^{(0,0)}, D^{(0,\pm 2)}] = \pm 2D^{(0,\pm 2)} , \quad [D^{(0,2)}, D^{(0,-2)}] = \tilde{D}^{(0,0)} . \quad (5.4b)$$

Any two operators from the different sets commute with each other.

We will work with matter multiplets $\Phi^{(p,q)}(z, u^\pm, v^\pm)$ that transform under the gauge group as in eq. (2.1). Since the gauge parameters $\tau(z)$ in (2.1) are harmonic independent, we now have a larger set of covariant derivatives

$$\mathcal{D}_{\underline{A}} = (\mathcal{D}_A, \mathcal{D}^{(\pm 2,0)}, \mathcal{D}^{(0,0)}, \mathcal{D}^{(0,\pm 2)}, \tilde{\mathcal{D}}^{(0,0)}) := (\mathcal{D}_A, D^{(\pm 2,0)}, D^{(0,0)}, D^{(0,\pm 2)}, \tilde{D}^{(0,0)}) \quad (5.5)$$

possessing the gauge transformation

$$\mathcal{D}'_{\underline{A}} = e^{i\tau} \mathcal{D}_{\underline{A}} e^{-i\tau} . \quad (5.6)$$

We introduce a new basis for the spinor gauge covariant derivatives³ $\mathcal{D}_\alpha^{\tilde{i}\tilde{j}}$ and the gauge covariant field strengths \mathcal{W}^{ij} and $\mathcal{W}^{\tilde{i}\tilde{j}}$ as follows:

$$\mathcal{D}_\alpha^{\tilde{i}\tilde{j}} \rightarrow \mathcal{D}_\alpha^{(\pm 1, \pm 1)} = u_i^\pm v_{\tilde{j}}^\pm \mathcal{D}_\alpha^{\tilde{i}\tilde{j}} , \quad (5.7a)$$

$$\mathcal{W}^{ij} \rightarrow (\mathcal{W}^{(2,0)}, \mathcal{W}^{(-2,0)}, \mathcal{W}^{(0,0)}) = (u_i^+ u_j^+, u_i^- u_j^-, u_i^+ u_j^-) \mathcal{W}^{ij} , \quad (5.7b)$$

$$\mathcal{W}^{\tilde{i}\tilde{j}} \rightarrow (\mathcal{W}^{(0,2)}, \mathcal{W}^{(0,-2)}, \tilde{\mathcal{W}}^{(0,0)}) = (v_i^+ v_{\tilde{j}}^+, v_i^- v_{\tilde{j}}^-, v_i^+ v_{\tilde{j}}^-) \mathcal{W}^{\tilde{i}\tilde{j}} . \quad (5.7c)$$

³Switching off the gauge multiplet in (5.7a) defines the new basis for the ordinary spinor covariant derivatives $D_\alpha^{\tilde{i}\tilde{j}}$.

Then the anti-commutation relation (2.2) leads to

$$\{\mathcal{D}_\alpha^{(1,1)}, \mathcal{D}_\beta^{(1,1)}\} = 0 , \quad (5.8a)$$

$$\{\mathcal{D}_\alpha^{(1,1)}, \mathcal{D}_\beta^{(-1,-1)}\} = 2i\mathcal{D}_{\alpha\beta} - \varepsilon_{\alpha\beta}\mathcal{W}^{(0,0)} - \varepsilon_{\alpha\beta}\tilde{\mathcal{W}}^{(0,0)} , \quad (5.8b)$$

$$\{\mathcal{D}_\alpha^{(1,-1)}, \mathcal{D}_\beta^{(-1,1)}\} = -2i\mathcal{D}_{\alpha\beta} + \varepsilon_{\alpha\beta}\mathcal{W}^{(0,0)} - \varepsilon_{\alpha\beta}\tilde{\mathcal{W}}^{(0,0)} , \quad (5.8c)$$

as well as to several additional relations which can be obtained from (5.8) by making use of the identities

$$[\mathcal{D}^{(2,0)}, \mathcal{D}_\alpha^{(1,\pm 1)}] = 0 , \quad [\mathcal{D}^{(0,2)}, \mathcal{D}_\alpha^{(\pm 1,1)}] = 0 , \quad (5.9a)$$

$$[\mathcal{D}^{(-2,0)}, \mathcal{D}_\alpha^{(1,\pm 1)}] = \mathcal{D}_\alpha^{(-1,\pm 1)} , \quad [\mathcal{D}^{(0,-2)}, \mathcal{D}_\alpha^{(\pm 1,1)}] = \mathcal{D}_\alpha^{(\pm 1,-1)} \quad (5.9b)$$

in conjunction with the relations

$$\mathcal{D}^{(2,0)}\mathcal{W}^{(0,0)} = \mathcal{W}^{(2,0)} , \quad \mathcal{D}^{(2,0)}\mathcal{W}^{(2,0)} = 0 , \quad (5.10a)$$

$$\mathcal{D}^{(0,2)}\tilde{\mathcal{W}}^{(0,0)} = \mathcal{W}^{(2,0)} , \quad \mathcal{D}^{(0,2)}\mathcal{W}^{(0,2)} = 0 . \quad (5.10b)$$

In particular, one obtains

$$\{\mathcal{D}_\alpha^{(1,1)}, \mathcal{D}_\beta^{(1,-1)}\} = -\varepsilon_{\alpha\beta}\mathcal{W}^{(2,0)} , \quad \{\mathcal{D}_\alpha^{(1,1)}, \mathcal{D}_\beta^{(-1,1)}\} = -\varepsilon_{\alpha\beta}\mathcal{W}^{(0,2)} , \quad (5.11a)$$

$$\{\mathcal{D}_\alpha^{(1,-1)}, \mathcal{D}_\beta^{(-1,-1)}\} = -\varepsilon_{\alpha\beta}\mathcal{W}^{(0,-2)} , \quad \{\mathcal{D}_\alpha^{(-1,1)}, \mathcal{D}_\beta^{(-1,-1)}\} = -\varepsilon_{\alpha\beta}\mathcal{W}^{(-2,0)} . \quad (5.11b)$$

The Bianchi identities (2.3) imply the analyticity constraints

$$\mathcal{D}_\alpha^{(1,\pm 1)}\mathcal{W}^{(2,0)} = 0 , \quad \mathcal{D}_\alpha^{(\pm 1,1)}\mathcal{W}^{(0,2)} = 0 . \quad (5.12)$$

Remarkably, all information about the field strengths is encoded in the following equations:

$$\mathcal{D}_\alpha^{(1,1)}\mathcal{W}^{(2,0)} = 0 , \quad \mathcal{D}^{(2,0)}\mathcal{W}^{(2,0)} = 0 , \quad \mathcal{D}^{(0,2)}\mathcal{W}^{(2,0)} = 0 ; \quad (5.13a)$$

$$\mathcal{D}_\alpha^{(1,1)}\mathcal{W}^{(0,2)} = 0 , \quad \mathcal{D}^{(2,0)}\mathcal{W}^{(0,2)} = 0 , \quad \mathcal{D}^{(0,2)}\mathcal{W}^{(0,2)} = 0 . \quad (5.13b)$$

Indeed, the third equation in (5.13a) tells us that $\mathcal{W}^{(2,0)}$ is independent of v^\pm , that is $\mathcal{W}^{(2,0)} = \mathcal{W}^{(2,0)}(z, u^\pm)$. The second equation in (5.13a) tells us that $\mathcal{W}^{(2,0)}$ is independent of the harmonics u_i^- and has the form (5.7b). Finally, the first equation in (5.13a) tells us that \mathcal{W}^{ij} obeys the Bianchi identity (2.3).

Eq. (5.8a) has two nontrivial implications. Firstly, it allows one to introduce *covariantly semi-analytic* superfields $\Phi^{(p,q)}(z, u^\pm, v^\pm)$ constrained by

$$\mathcal{D}_\alpha^{(1,1)}\Phi^{(p,q)} = 0 . \quad (5.14)$$

Such multiplets are rigid-superspace analogues of the covariant hybrid multiplets introduced in [18] in the framework of $\mathcal{N} = 4$ supergravity. Secondly, the constraint (5.8a) has the following general solution

$$\mathcal{D}_\alpha^{(1,1)} = e^{-i\Omega} D_\alpha^{(1,1)} e^{i\Omega} , \quad \Omega \equiv \Omega^{(0,0)} , \quad (5.15)$$

for some bridge superfield $\Omega(z, u^\pm, v^\pm)$ which takes its values in the Lie algebra of the gauge group.

Switching off the vector multiplet in (5.14) defines semi-analytic superfields,

$$D_\alpha^{(1,1)} \phi^{(p,q)} = 0 . \quad (5.16)$$

In complete analogy with the harmonic superspace approach [27, 28], for such multiplets one can define a modified conjugation that maps every semi-analytic superfield $\phi^{(p,q)}(z, u^\pm, v^\pm)$ into a semi-analytic one, $\check{\phi}^{(p,q)}(z, u^\pm, v^\pm)$, of the same $U(1)$ charge. We will refer to it as “smile-conjugation.” The smile-conjugations has the property

$$\check{\check{\phi}}^{(p,q)}(z, u^\pm, v^\pm) = (-1)^{p+q} \phi^{(p,q)}(z, u^\pm, v^\pm) . \quad (5.17)$$

Thus in the case that $(p+q)$ is even, real semi-analytic superfields may be introduced.

The introduction of Ω leads to a new gauge freedom, in addition to the τ gauge symmetry (2.1). The gauge transformation of Ω is

$$e^{i\Omega'} = e^{i\lambda} e^{i\Omega} e^{-i\tau} , \quad \lambda \equiv \lambda^{(0,0)} , \quad (5.18a)$$

where the new gauge parameter $\lambda(z, u^\pm, v^\pm)$ is constrained to be semi-analytic,

$$D_\alpha^{(1,1)} \lambda = 0 , \quad (5.18b)$$

and is real with respect to the smile-conjugation. The bridge Ω in (5.15) may be chosen Hermitian with respect to the operations of transposition and smile-conjugation.

Making use of the bridge allows us to introduce a new representation for the covariant derivatives $\mathcal{D}_\underline{A}$ and matter multiplets $\Phi^{(p,q)}$ with the property that no τ -gauge freedom is left. It is obtained by applying the transformation:

$$\mathcal{D}_\underline{A} \rightarrow \nabla_\underline{A} = e^{i\Omega} \mathcal{D}_\underline{A} e^{-i\Omega} , \quad (5.19a)$$

$$\Phi^{(p,q)} \rightarrow \phi^{(p,q)} = e^{i\Omega} \Phi^{(p,q)} , \quad (5.19b)$$

in particular

$$\mathcal{W}^{(p,q)} \rightarrow W^{(p,q)} = e^{i\Omega} \mathcal{W}^{(p,q)} e^{-i\Omega} . \quad (5.19c)$$

for the field strengths (5.7b) and (5.7c). The resulting λ -representation is characterised by two important properties. Firstly, the covariant derivative $\nabla_\alpha^{(1,1)}$ has no gauge connection,

$$\nabla_\alpha^{(1,1)} = D_\alpha^{(1,1)} . \quad (5.20)$$

Secondly, the harmonic covariant derivatives acquire gauge connections,

$$\nabla^{(\pm 2,0)} = D^{(\pm 2,0)} + iV^{(\pm 2,0)} , \quad \nabla^{(0,\pm 2)} = D^{(0,\pm 2)} + iV^{(0,\pm 2)} . \quad (5.21)$$

Under the λ -gauge transformation, the gauge connections in (5.21) change as

$$iV'^{(\pm 2,0)} = e^{i\lambda}(\nabla^{(\pm 2,0)}e^{-i\lambda}) , \quad iV'^{(0,\pm 2)} = e^{i\lambda}(\nabla^{(0,\pm 2)}e^{-i\lambda}) . \quad (5.22)$$

In the λ -representation, the equations (5.9a) mean that the gauge prepotentials $V^{(2,0)}$ and $V^{(0,2)}$ are semi-analytic,

$$D_\alpha^{(1,1)}V^{(2,0)} = 0 , \quad D_\alpha^{(1,1)}V^{(0,2)} = 0 . \quad (5.23)$$

The above consideration in this section concerns the long vector multiplet. As discussed in section 2, the left and the right vector multiplets are obtained from the long one by imposing the additional constraints (2.4a) and (2.4b), respectively. This leads to important specific features, which we now analyse. It suffices to consider only the left multiplet for which $\mathcal{W}^{\tilde{i}\tilde{j}} = 0$, and hence $\mathcal{W}^{(0,2)} = \mathcal{W}^{(0,-2)} = \tilde{\mathcal{W}}^{(0,0)} = 0$; the case of the right vector multiplet is analogous. Since the right-hand sides of eqs. (5.8) and (5.11) are independent of the v^\pm harmonics, the bridge Ω in (5.15) can also be chosen to be independent of these harmonics, $\Omega = \Omega(z, u^\pm)$. The gauge parameter λ in (5.18) also becomes independent of the v^\pm harmonics, $\lambda = \lambda(z, u^\pm)$, and the semi-analyticity constraint (5.18b) turns into the analyticity conditions $D_\alpha^{(1,\pm 1)}\lambda = 0$. These results have two important corollaries: (i) the harmonic connections $V^{(0,\pm 2)}$ in (5.21) vanish, $V^{(0,\pm 2)} = 0$; (ii) the connections $V^{(\pm 2,0)}$ are independent of the v^\pm harmonics, $V^{(\pm 2,0)} = V^{(\pm 2,0)}(z, u^\pm)$. As a result, the first equation in (5.23) obeys the stronger analyticity conditions $D_\alpha^{(1,\pm 1)}V^{(2,0)} = 0$, which agrees with Zupnik's approach [13, 14].

The zero-curvature conditions (5.4a) and (5.4b) in the λ -frame are

$$D^{(2,0)}V^{(-2,0)} - D^{(-2,0)}V^{(2,0)} + i[V^{(2,0)}, V^{(-2,0)}] = 0 , \quad (5.24a)$$

$$D^{(0,2)}V^{(0,-2)} - D^{(0,-2)}V^{(0,2)} + i[V^{(0,2)}, V^{(0,-2)}] = 0 . \quad (5.24b)$$

They allow one to express the superfields $V^{(-2,0)}$ and $V^{(0,-2)}$ in terms of $V^{(2,0)}$ and $V^{(0,2)}$, respectively,

$$V^{(-2,0)} = V^{(-2,0)}[V^{(2,0)}] , \quad V^{(0,-2)} = V^{(0,-2)}[V^{(0,2)}] . \quad (5.25)$$

Explicitly, these solutions are given as series over harmonic distributions presented in [29]. We point out that the superfields $V^{(-2,0)}$ and $V^{(0,-2)}$ live in the full superspace in contrast to the prepotentials $V^{(2,0)}$ and $V^{(0,2)}$ subject to the constraints (5.23).

In the case of the left vector multiplet, for which $\mathcal{W}^{\tilde{i}\tilde{j}} = 0$ and $V^{(0,\pm 2)} = 0$, the prepotential $V^{(2,0)}$ is real analytic but otherwise unconstrained. For the long vector multiplet, we have two semi-analytic prepotentials $V^{(2,0)}$ and $V^{(0,2)}$. The constraints (5.23) are not the only conditions they obey. They are also related to each other by the zero-curvature condition

$$D^{(2,0)}V^{(0,2)} - D^{(0,2)}V^{(2,0)} + i[V^{(2,0)}, V^{(0,2)}] = 0 . \quad (5.26)$$

Using the algebra of spinor covariant derivatives in the λ -frame it is possible to express the field strengths $W^{(2,0)}$ and $W^{(0,2)}$ in terms of the gauge prepotentials $V^{(0,-2)}$ and $V^{(-2,0)}$. The resulting expressions are

$$W^{(2,0)} = \frac{i}{2}D^{(1,1)\alpha}D_{\alpha}^{(1,1)}V^{(0,-2)} , \quad W^{(0,2)} = \frac{i}{2}D^{(1,1)\alpha}D_{\alpha}^{(1,1)}V^{(-2,0)} . \quad (5.27)$$

These superfields transform covariantly under the λ -gauge group,

$$W'^{(2,0)} = e^{i\lambda}W^{(2,0)}e^{-i\lambda} , \quad W'^{(0,2)} = e^{i\lambda}W^{(0,2)}e^{-i\lambda} . \quad (5.28)$$

These results show that the long vector multiplet is completely described in terms of the two real semi-analytic prepotentials $V^{(2,0)}$ and $V^{(0,2)}$ subject to the zero-curvature condition (5.26).

As discussed above, all information about the off-shell long vector multiplet is encoded in the equations (5.13). It is interesting that the on-shell hypermultiplets are described by analogous equations:

$$\mathcal{D}_{\alpha}^{(1,1)}q^{(1,0)} = 0 , \quad \mathcal{D}^{(2,0)}q^{(1,0)} = 0 , \quad \mathcal{D}^{(0,2)}q^{(1,0)} = 0 ; \quad (5.29a)$$

$$\mathcal{D}_{\alpha}^{(1,1)}q^{(0,1)} = 0 , \quad \mathcal{D}^{(2,0)}q^{(0,1)} = 0 , \quad \mathcal{D}^{(0,2)}q^{(0,1)} = 0 . \quad (5.29b)$$

For example, consider the left hypermultiplet. The third equation in (5.29a) tells us that $q^{(1,0)}$ is independent of v^{\pm} , that is $q^{(1,0)} = q^{(1,0)}(z, u^{\pm})$. The second equation in (5.29a) tells us that $q^{(1,0)}$ is independent of the harmonics u_i^{-} and has the functional form $q^{(1,0)} = u_i^{+}q^i$. Finally, the first equation in (5.29a) tells us that q^i obeys the constraint (2.5a). In summary, the on-shell hypermultiplets in harmonic superspace are described by the superfields

$$q^{(1,0)} = u_i^{+}q^i , \quad q^{(0,1)} = v_i^{+}\tilde{q}^{\tilde{i}} . \quad (5.30)$$

In conclusion of this section let us briefly discuss the equations of motion for the ABJM theory in $\mathcal{N} = 4$ harmonic superspace. This theory is described by the hypermultiplet superfields $q^{(0,1)}$ and $q^{(1,0)}$ in the bi-fundamental representation of the gauge group $G_{\mathfrak{L}} \times G_{\mathfrak{R}}$. There are also superfield strengths $W_{\mathfrak{L}}^{(2,0)}$, $W_{\mathfrak{R}}^{(2,0)}$, $W_{\mathfrak{L}}^{(0,2)}$ and $W_{\mathfrak{R}}^{(0,2)}$ which take values in the Lie algebras of the gauge groups $G_{\mathfrak{L}}$ and $G_{\mathfrak{R}}$. These superfields obey the equations (5.13) and (5.29). The field strengths are expressed in terms of the hypermultiplets as

$$\begin{aligned} W_{\mathfrak{L}}^{(2,0)} &= i\kappa q^{(1,0)} \bar{q}^{(1,0)} , & W_{\mathfrak{R}}^{(2,0)} &= i\kappa \bar{q}^{(1,0)} q^{(1,0)} , \\ W_{\mathfrak{L}}^{(0,2)} &= i\kappa q^{(0,1)} \bar{q}^{(0,1)} , & W_{\mathfrak{R}}^{(0,2)} &= i\kappa \bar{q}^{(0,1)} q^{(0,1)} . \end{aligned} \quad (5.31)$$

It would be interesting to find a superfield Lagrangian reproducing this set of equations.

6 Conclusion

We have shown that the $\mathcal{N} = 4$ superfield realisations for superconformal CS theories with $\mathcal{N} \geq 4$ require the long $\mathcal{N} = 4$ vector multiplet. The structure of the long $\mathcal{N} = 4$ vector multiplet turns out to be the main reason for problems with constructing off-shell actions in $\mathcal{N} = 4$ superspace for supersymmetric CS theories with eight and more supercharges. The simplest way to see this is to look at $\mathcal{N} = 4 \rightarrow \mathcal{N} = 3$ superspace reduction of large $\mathcal{N} = 4$ vector multiplet. In $\mathcal{N} = 3$ superspace, this multiplet is described by gauge covariant symmetric isospinors \mathbb{W}^{ij} and $\tilde{\mathbb{W}}^{ij}$ in the adjoint representation of the gauge group. One of them, \mathbb{W}^{ij} , is the field strength of the $\mathcal{N} = 3$ vector multiplet [30, 13]. In terms of the gauge covariant derivatives $\mathcal{D}_{\alpha}^{ij}$ in $\mathcal{N} = 3$ superspace, it originates as follows

$$\{\mathcal{D}_{\alpha}^{ij}, \mathcal{D}_{\beta}^{kl}\} = -2i\varepsilon^{i(k}\varepsilon^{l)j}\mathcal{D}_{\alpha\beta} + \frac{1}{2}\varepsilon_{\alpha\beta}(\varepsilon^{i(k}\mathbb{W}^{l)j} + \varepsilon^{j(k}\mathbb{W}^{l)i}) \quad (6.1)$$

and obeys the Bianchi identity

$$\mathcal{D}_{\alpha}^{(ij}\mathbb{W}^{kl)} = 0 . \quad (6.2)$$

The other object, $\tilde{\mathbb{W}}^{ij}$, is a Lie-algebra-valued matter multiplet subject to the same constraint as \mathbb{W}^{ij} ,

$$\mathcal{D}_{\alpha}^{(ij}\tilde{\mathbb{W}}^{kl)} = 0 . \quad (6.3)$$

Each of \mathbb{W}^{ij} and $\tilde{\mathbb{W}}^{ij}$ is a linear combination of the $\mathcal{N} = 4$ field strengths \mathcal{W}^{ij} and $\mathcal{W}^{\tilde{i}\tilde{j}}$ in (2.2) projected to $\mathcal{N} = 3$ superspace. On the mass shell, \mathbb{W}^{ij} and $\tilde{\mathbb{W}}^{ij}$

become composites constructed from $\mathcal{N} = 3$ hypermultiplets q^i and their conjugates \bar{q}_i . Symbolically, we have $\mathbb{W}_A^{ij} = \bar{q}^{(i} T_A q^{j)}$ and $\tilde{\mathbb{W}}_A^{ij} = \bar{q}^{(i} \tilde{T}_A q^{j)}$. The former equation can always be realised as the equation of motion for the vector multiplet in some $\mathcal{N} = 3$ superconformal Chern-Simons-matter theory formulated in harmonic superspace [10]. However, there is no systematic way to realise the latter constraint as an Euler-Lagrange equation, except for the abelian case.

The results of this paper can naturally be generalised to supergravity. The equations of motion for a general $\mathcal{N} = 4$ superconformal CS theory coupled to $\mathcal{N} = 4$ conformal supergravity are

$$\nabla_\alpha^{\tilde{i}(i} q^{j)} = 0, \quad \nabla_\alpha^{i(\tilde{i}} q^{\tilde{j})} = 0, \quad (6.4a)$$

$$\frac{1}{\kappa} \mathcal{W}_A^{ij} = i g_{AB} \bar{q}^{(i} T^B q^{j)}, \quad \frac{1}{\tilde{\kappa}} \tilde{\mathcal{W}}_A^{\tilde{i}\tilde{j}} = i g_{AB} \bar{q}^{(\tilde{i}} \tilde{T}^B q^{\tilde{j})}, \quad (6.4b)$$

$$\frac{1}{\kappa_{\text{SG}}} W = \bar{q}_i q^i - \bar{\tilde{q}}_{\tilde{i}} \tilde{q}^{\tilde{i}}, \quad (6.4c)$$

where W is the $\mathcal{N} = 4$ super-Cotton scalar (see [25] for more details). The torsion and curvature tensors in $\mathcal{N} = 4$ conformal superspace are completely determined in terms of W and its covariant derivatives. The super-Cotton scalar obeys the Bianchi identity [25]

$$\nabla^\alpha{}^{(\tilde{i}(i} \nabla_\alpha^{j)\tilde{j})} W = 0, \quad (6.5)$$

and the same equation is obeyed by each term on the right of (6.4c). The gauge-covariant derivative ∇ in eq. (6.4a) is defined in Appendix B, eq. (B.1). In this paragraph as well as in Appendix B, we use the notation $\nabla_{\mathcal{A}}$ for the covariant derivatives in conformal superspace. These should not be confused with the gauge covariant derivative in the λ -frame (5.19).

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A Consistency condition in $\mathcal{N} = 4$ CS theories

As shown in section 2, the $\mathcal{N} = 4$ supersymmetric gauge theories are subject to the consistency conditions (2.6). In the case of superconformal CS theories, these

conditions imply the equations (2.9) for the generators of the gauge group. Here we demonstrate that these equations are equivalent to the fundamental identity for the generators of the gauge group found in [8]. For simplicity, here we consider only the left hypermultiplet q^i and field strength \mathcal{W}^{ij} ; the analysis for the right multiplets $\tilde{q}^{\tilde{i}}$ and $\mathcal{W}^{\tilde{i}\tilde{j}}$ is absolutely identical.

We can view the hypermultiplet $(q^i) = q_a^i$, $a = 1, \dots, n$, as a n -vector in some representation of the gauge group G so that $(\bar{q}^i) = \bar{q}^{ia}$ form the conjugated representation. It is convenient to combine them into one $2n$ -dimensional vector $Q^i = (Q_{\hat{a}}^i) = (q_a^i, -\bar{q}^{ia})^T$, where $\hat{a} = 1, \dots, 2n$, corresponds to the $\mathbf{Sp}(2n)$ group such that

$$\bar{Q}_{\hat{a}}^i := \overline{Q_{\hat{a}}^i} = \varepsilon_{ij} \Omega^{\hat{a}\hat{b}} Q_{\hat{b}}^j, \quad \bar{Q}_i = (\bar{Q}_{\hat{a}}^i) = (\bar{q}_i^a, q_{ia}), \quad (\text{A.1})$$

where $\Omega^{\hat{a}\hat{b}} = -\Omega^{\hat{b}\hat{a}}$ is the invariant tensor of $\mathbf{Sp}(2n)$,

$$\Omega_{\hat{a}\hat{b}} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}. \quad (\text{A.2})$$

The gauge group G acts on Q^i by symplectic transformations. Denoting by $(\mathbf{T}^A)_{\hat{a}}^{\hat{b}}$ the generators of G in which $Q_{\hat{a}}^i$ transforms, the field strength is

$$\mathcal{W}^{ij} = \mathcal{W}_A^{ij} \mathbf{T}^A. \quad (\text{A.3})$$

In this notation, the integrability condition (2.6) can be rewritten as

$$\mathcal{W}_A^{(ij} (\mathbf{T}^A)_{\hat{a}}^{\hat{b}} Q_{\hat{b}}^{k)} = 0. \quad (\text{A.4})$$

In the superconformal CS theories considered in section 2, the field strength \mathcal{W}_A^{ij} becomes the hypermultiplet composite operator given by (2.7a). In the notation (A.1), eq. (2.7a) reads

$$\mathcal{W}_A^{ij} = \frac{i}{2} \kappa g_{AB} Q^{\hat{a}(i} (\mathbf{T}^B)_{\hat{a}}^{\hat{b}} Q_{\hat{b}}^{j)}, \quad (\text{A.5})$$

where the generator $(\mathbf{T}^A)_{\hat{a}}^{\hat{b}}$ has the following block diagonal form

$$(\mathbf{T}^A)_{\hat{a}}^{\hat{b}} = \begin{pmatrix} (T^A)_a^b & 0 \\ 0 & -(T^A)_b^a \end{pmatrix}. \quad (\text{A.6})$$

The consistency condition (A.4) now reads

$$g_{AB} Q^{\hat{a}(i} Q_{\hat{b}}^j Q_{\hat{d}}^k (\mathbf{T}^A)_{\hat{a}}^{\hat{b}} (\mathbf{T}^B)_{\hat{c}}^{\hat{d}} = 0. \quad (\text{A.7})$$

This equation implies the following constraint on the generators of the gauge group

$$g_{AB} \mathbf{T}_{\hat{a}(\hat{b}}^A \mathbf{T}_{\hat{c}\hat{d})}^B = 0, \quad (\text{A.8})$$

where we have assumed that the hypermultiplet indices, \hat{a}, \hat{b}, \dots , are raised and lowered using the symplectic metric $\Omega^{\hat{a}\hat{b}}$ and its inverse $\Omega_{\hat{a}\hat{b}}$, $\Omega_{\hat{a}\hat{b}}\Omega^{\hat{b}\hat{c}} = \delta_{\hat{a}}^{\hat{c}}$. For the generators with lower indices, $\mathbf{T}_{\hat{a}\hat{b}}^A = \Omega_{\hat{b}\hat{c}}(\mathbf{T}^A)_{\hat{a}}^{\hat{c}}$, we have

$$(\mathbf{T}^A)_{\hat{a}\hat{b}} = \begin{pmatrix} 0 & (T^A)_a{}^b \\ (T^A)_b{}^a & 0 \end{pmatrix}. \quad (\text{A.9})$$

With this block matrix representation of the generators it becomes obvious that eq. (A.8) is equivalent to (2.9a),

$$g_{AB}\mathbf{T}_{\hat{a}(\hat{b}}^A\mathbf{T}_{\hat{c}\hat{d})}^B = 0 \quad \Longleftrightarrow \quad g_{AB}(T^A)_a{}^{(b}(T^B)_{c}{}^{d)} = 0. \quad (\text{A.10})$$

The equation (A.8) was first derived in [8] where the formulation in terms of $\mathcal{N} = 1$ superfields was developed for $\mathcal{N} = 4$ superconformal CS theories. Eq. (A.8) was necessary for the construction of consistent interaction Lagrangians for $\mathcal{N} = 4$ superconformal CS theories. As demonstrated in our paper, in $\mathcal{N} = 4$ superspace eq. (A.8) naturally arises as the consistency condition of the hypermultiplet equations of motion.

In [8], the equation (A.8) was named as the *fundamental identity* since it imposes non-trivial constraints both on the gauge group and the matter representation. In the same paper it was demonstrated that the fundamental identity is satisfied for those Lie groups which allow for super-extensions. The typical examples of such gauge groups are $\text{U}(M) \times \text{U}(N)$ and $\text{O}(M) \times \text{Sp}(2N)$ which are the bosonic bodies of $\text{U}(M|N)$ and $\text{OSp}(M|2N)$, respectively. The matter hypermultiplets belong to the bi-fundamental representations of these gauge groups.

B \mathcal{N} -extended vector multiplet

To describe a Yang-Mills multiplet in the 3D \mathcal{N} -extended conformal superspace $\mathcal{M}^{3|2\mathcal{N}}$ of [25], parametrized by coordinates $z^M = (x^m, \theta_I^\mu)$, we introduce gauge covariant derivatives

$$\nabla_{\mathcal{A}} = (\nabla_a, \nabla_\alpha^I) := \nabla_{\mathcal{A}} + iV_{\mathcal{A}}, \quad I = 1, \dots, \mathcal{N}, \quad (\text{B.1})$$

where $\nabla_{\mathcal{A}}$ are the supergravity covariant derivatives [25]. The algebra of gauge covariant derivative is

$$[\nabla_{\mathcal{A}}, \nabla_{\mathcal{B}}] = i\mathcal{F}_{\mathcal{AB}} + \dots, \quad (\text{B.2})$$

where the ellipsis denotes the purely supergravity terms. The field strength \mathcal{F}_{AB} satisfies the Bianchi identity

$$\nabla_{[A}\mathcal{F}_{BC]} + T_{[AB}{}^D\mathcal{F}_{D|C]} = 0 , \quad (\text{B.3})$$

where $T_{AB}{}^D$ is the torsion tensor, see [25] for more details. The field strength is subject to a covariant constraint to describe a vector multiplet. For $\mathcal{N} > 1$ the constraint [31, 32, 30] is

$$\mathcal{F}_{\alpha\beta}^{IJ} = 2i\varepsilon_{\alpha\beta}\mathcal{W}^{IJ} , \quad (\text{B.4})$$

The Bianchi identities then give the remaining components of the field strength [25]:

$$\mathcal{F}_{a\alpha}^I = -\frac{1}{(\mathcal{N}-1)}(\gamma_a)_\alpha{}^\beta\nabla_{\beta J}\mathcal{W}^{IJ} , \quad (\text{B.5a})$$

$$\mathcal{F}_{ab} = \frac{i}{4\mathcal{N}(\mathcal{N}-1)}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}[\nabla_\alpha^K, \nabla_\beta^L]\mathcal{W}_{KL} . \quad (\text{B.5b})$$

For $\mathcal{N} > 2$ the field strength \mathcal{W}^{IJ} is constrained by the dimension-3/2 Bianchi identity

$$\nabla_\gamma^I\mathcal{W}^{JK} = \nabla_\gamma^{[I}\mathcal{W}^{JK]} - \frac{2}{\mathcal{N}-1}\delta^{I[J}\nabla_{\gamma L}\mathcal{W}^{K]L} . \quad (\text{B.6})$$

This constraint may be shown to define an off-shell supermultiplet [21], see also [25].

The component fields of vector multiplets may be extracted from the field strength \mathcal{W}^{IJ} . For $\mathcal{N} > 1$, we define the matter fields as follows

$$w^{IJ} := \mathcal{W}^{IJ}| , \quad (\text{B.7a})$$

$$\lambda_\alpha^I := \frac{2}{\mathcal{N}-1}\nabla_{\alpha J}\mathcal{W}^{IJ}| , \quad (\text{B.7b})$$

$$h^{IJ} := \frac{i}{\mathcal{N}-1}\nabla^{\gamma[I}\nabla_{\gamma K}\mathcal{W}^{J]K}| , \quad (\text{B.7c})$$

$$\chi_{\alpha_1\cdots\alpha_n}{}^{I_1\cdots I_{n+2}} := I(n)\nabla_{(\alpha_1}^{I_1}\cdots\nabla_{\alpha_n}^{I_n}\mathcal{W}^{I_{n+1}I_{n+2}}| , \quad (\text{B.7d})$$

where

$$I(n) = \begin{cases} i , & n = 1, 2 \pmod{4} \\ 1 , & n = 3, 4 \pmod{4} . \end{cases} \quad (\text{B.8})$$

For the supersymmetry transformations of the \mathcal{N} -extended vector multiplet, see [11].

The component fields of the vector multiplet form the following tower [21, 11]:

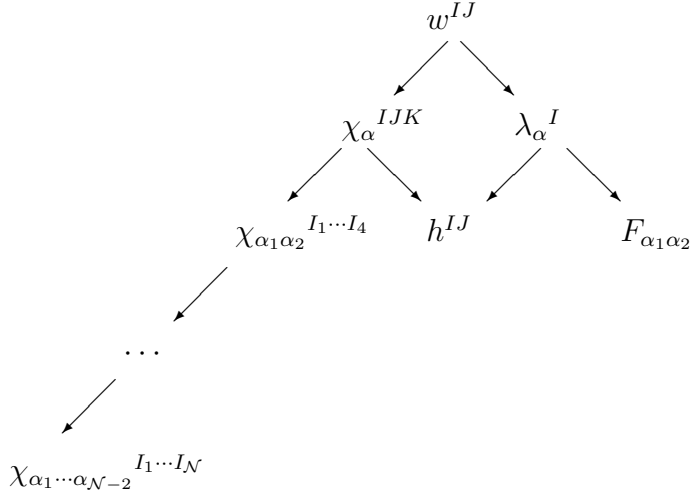


Figure 1. Component fields of the \mathcal{N} -extended vector multiplet

Here $F_{\alpha\beta}$ is the symmetric spinor associated with the Hodge dual $F^a = \frac{1}{2}\varepsilon^{abc}F_{bc}$ of $F_{ab} = \mathcal{F}_{ab}$. Modulo fermionic terms, F_{ab} coincides with the component field strength. In the left branch of the diagram, the fields $\chi_{\alpha_1\alpha_2}^{I_1\cdots I_4}, \dots, \chi_{\alpha_1\cdots\alpha_{N-2}}^{I_1\cdots I_N}$ satisfy, in the linearised approximation, the conservation equations $(\gamma^a)^{\beta\gamma}\partial_a\chi_{\beta\gamma\alpha_3\cdots\alpha_n}^{I_1\cdots I_{n+2}} = 0$.

In the $\mathcal{N} = 4$ case, the two branches of the diagram have identical algebraic structure, and thus every field on the left has a twin on the right. This doubling of fields disappears if the field strength \mathcal{W}^{IJ} is constrained to be self-dual, $\tilde{\mathcal{W}}^{IJ} = \mathcal{W}^{IJ}$, or anti-self-dual, $\tilde{\mathcal{W}}^{IJ} = -\mathcal{W}^{IJ}$, where

$$\tilde{\mathcal{W}}^{IJ} := \frac{1}{2}\varepsilon^{IJKL}\mathcal{W}_{KL} . \quad (\text{B.9})$$

These cases correspond to the left and right vector multiplets, respectively.

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